

Arithmetic formulas for the Fourier coefficients of Hauptmodules of level 2, 3, and 5

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Abstract

We give arithmetic formulas for the coefficients of Hauptmodules of higher level as analogues of Kaneko's result. We also obtain their asymptotic formulas by employing Murty-Sampanth's method.

1 Introduction

In [1], Kaneko gave the following arithmetic formula for the Fourier coefficients of the elliptic modular function $j(\tau)$. Let $\mathbf{t}_m(d)$ be the modular trace function (the precise definition will be given later) and c_n ($n \geq 1$) the n th Fourier coefficient of $j(\tau)$, that is, $j(\tau) = q^{-1} + 744 + \sum_{n=1}^{\infty} c_n q^n$. Then the formula for c_n is given by

$$\begin{aligned} c_n &= \frac{1}{n} \left\{ \sum_{r \in \mathbb{Z}} \mathbf{t}_1(n - r^2) + \sum_{\substack{r \geq 1 \\ r: \text{odd}}} ((-1)^n \mathbf{t}_1(4n - r^2) - \mathbf{t}_1(16n - r^2)) \right\} \\ &= \frac{1}{2n} \sum_{r \in \mathbb{Z}} \mathbf{t}_2(4n - r^2). \end{aligned}$$

On the other hand, by using the circle method, Petersson [2] and later Rademacher [3] independently derived the asymptotic formula for c_n :

$$c_n \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}} \text{ as } n \rightarrow \infty.$$

Recently, Murty and Sampanth [4] proved this formula by using the above arithmetic formula and Laplace's method instead of the circle method.

In this article, we generalize these formulas to Hauptmodules for the congruence subgroups $\Gamma_0(p)$ and $\Gamma_0^*(p)$ (the extension of $\Gamma_0(p)$ by the Atkin-Lehner involution) with $p = 2, 3$, and 5.

Let $j_p(\tau)$ and $j_p^*(\tau)$ be the corresponding Hauptmodules for $\Gamma_0(p)$ and $\Gamma_0^*(p)$, respectively. Ohta [5] gave the arithmetic formulas for the Fourier coefficients of $j_2(\tau)$ and $j_2^*(\tau)$, and a part of those of $j_3(\tau)$. She also treated the cases of $j_4(\tau)$ and $j_4^*(\tau)$. Let $c_n^{(p)}$ and $c_n^{(p*)}$ be the n th Fourier coefficient of $j_p(\tau)$ and $j_p^*(\tau)$, respectively. These coefficients can be expressed in terms of the modular trace functions $\mathbf{t}_m^{(p*)}(d)$.

Theorem 1.1. *For any $n \geq 1$, we have*

$$\begin{aligned} c_n^{(2)} &= \frac{1}{2n} \times \begin{cases} -\sum_{r \equiv 0(2)} \mathbf{t}_2^{(2*)}(4n - r^2) + 24\sigma_1^{(2)}(n) & (n \equiv 0 \pmod{2}), \\ \sum_{r \in \mathbb{Z}} \mathbf{t}_2^{(2*)}(4n - r^2) + 24\sigma_1(n) & (n \not\equiv 0 \pmod{2}), \end{cases} \\ c_n^{(3)} &= \frac{1}{2n} \times \begin{cases} -\sum_{r \equiv 0(3)} \mathbf{t}_2^{(3*)}(4n - r^2) + 36\sigma_1^{(3)}(n) & (n \equiv 0 \pmod{3}), \\ \sum_{r \in \mathbb{Z}} \mathbf{t}_2^{(3*)}(4n - r^2) + 36\sigma_1(n) & (n \not\equiv 0 \pmod{3}), \end{cases} \\ c_n^{(5)} &= \frac{1}{2n} \times \begin{cases} -\sum_{r \equiv 0(5)} \mathbf{t}_2^{(5*)}(4n - r^2) + 18\sigma_1^{(5)}(n) & (n \equiv 0 \pmod{5}), \\ \sum_{r \in \mathbb{Z}} \mathbf{t}_2^{(5*)}(4n - r^2) + 18\sigma_1(n) & (n \not\equiv 0 \pmod{5}), \end{cases} \\ c_n^{(p*)} &= c_n^{(p)} - pc_{pn}^{(p)} \quad (p = 2, 3, 5) \end{aligned}$$

where $\sigma_1(n) = \sum_{d|n} d$, and $\sigma_1^{(p)}(n) = \sum_{\substack{d|n \\ (d,p)=1}} d$.

Remark 1.2. These formulas are different from those in Ohta [5]. In [5], the definition of $t_m^{(p)}(d)$ was mixed with that of $t_m^{(p*)}(d)$, and used the values of $t_m^{(p)}(d)$ instead of $t_m^{(p*)}(d)$.

Combining these formulas with Laplace's method as in [4], we obtain the asymptotic formulas of $c_n^{(p)}$.

Theorem 1.3. We have

$$\begin{aligned} c_n^{(2)} &\sim \frac{e^{2\pi\sqrt{n}}}{2n^{3/4}} \times \begin{cases} -1 & (n \equiv 0 \pmod{2}), \\ 1 & (n \equiv 1 \pmod{2}), \end{cases} \\ c_n^{(3)} &\sim \frac{e^{4\pi\sqrt{n}/3}}{\sqrt{6}n^{3/4}} \times \begin{cases} -1 & (n \equiv 0, 2 \pmod{3}), \\ 2 & (n \equiv 1 \pmod{3}), \end{cases} \\ c_n^{(5)} &\sim \frac{e^{4\pi\sqrt{n}/5}}{\sqrt{10}n^{3/4}} \times \begin{cases} -1 & (n \equiv 0 \pmod{5}), \\ (3 + \sqrt{5})/2 & (n \equiv 1 \pmod{5}), \\ -1 + \sqrt{5} & (n \equiv 2 \pmod{5}), \\ -1 - \sqrt{5} & (n \equiv 3 \pmod{5}), \\ (3 - \sqrt{5})/2 & (n \equiv 4 \pmod{5}) \end{cases} \end{aligned}$$

as $n \rightarrow \infty$.

2 Preliminaries

In this section, we shall define the Hauptmodules and the modular trace functions.

Definition 2.1. Let Γ be a congruence subgroup of $SL_2(\mathbb{R})/\pm I$ containing $(\begin{smallmatrix} 1 & \\ 0 & 1 \end{smallmatrix})$. If the genus of Γ is equal to 0, there is a unique modular function f of weight 0 satisfying the following conditions. We call this f the Hauptmodule with respect to Γ .

- (1) f is holomorphic in the upper half plane \mathfrak{H} ,
- (2) f has a Fourier expansion of the form $f(\tau) = q^{-1} + \sum_{n=1}^{\infty} H_n q^n$ ($q := e^{2\pi i\tau}$),
- (3) f is holomorphic at cusps of Γ except $i\infty$.

For $\Gamma_0(p) := \{(\begin{smallmatrix} a & \\ c & d \end{smallmatrix}) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{p}\}$ and $\Gamma_0^*(p) := \Gamma_0(p) \cup \Gamma_0(p)(\begin{smallmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{smallmatrix})$ ($p = 2, 3, 5$), the corresponding Hauptmodules $j_p(\tau)$ and $j_p^*(\tau)$ can be described by means of the Dedekind η -function $\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$;

$$\begin{aligned} j_2(\tau) &= \left(\frac{\eta(\tau)}{\eta(2\tau)} \right)^{24} + 24 = \frac{1}{q} + 276q - 2048q^2 + 11202q^3 + \cdots, \\ j_2^*(\tau) &= \left(\frac{\eta(\tau)}{\eta(2\tau)} \right)^{24} + 24 + 2^{12} \left(\frac{\eta(2\tau)}{\eta(\tau)} \right)^{24} = \frac{1}{q} + 4372q + 96256q^2 + 1240002q^3 + \cdots, \\ j_3(\tau) &= \left(\frac{\eta(\tau)}{\eta(3\tau)} \right)^{12} + 12 = \frac{1}{q} + 54q - 76q^2 - 243q^3 + \cdots, \\ j_3^*(\tau) &= \left(\frac{\eta(\tau)}{\eta(3\tau)} \right)^{12} + 12 + 3^6 \left(\frac{\eta(3\tau)}{\eta(\tau)} \right)^{12} = \frac{1}{q} + 783q + 8672q^2 + 65367q^3 + \cdots, \\ j_5(\tau) &= \left(\frac{\eta(\tau)}{\eta(5\tau)} \right)^6 + 6 = \frac{1}{q} + 9q + 10q^2 - 30q^3 + \cdots, \\ j_5^*(\tau) &= \left(\frac{\eta(\tau)}{\eta(5\tau)} \right)^6 + 6 + 5^3 \left(\frac{\eta(5\tau)}{\eta(\tau)} \right)^6 = \frac{1}{q} + 134q + 760q^2 + 3345q^3 + \cdots. \end{aligned}$$

For $p = 2, 3$, and 5 , let d be a positive integer such that $-d$ is congruent to a square modulo $4p$, and $\mathcal{Q}_{d,p}$ the set of positive definite binary quadratic forms $Q(X, Y) = [a, b, c] = aX^2 + bXY + cY^2$ ($a, b, c \in \mathbb{Z}$)

of discriminant $-d$ with $a \equiv 0 \pmod{p}$. Moreover, we fix an integer $\beta \pmod{2p}$ with $\beta^2 \equiv -d \pmod{4p}$ and denote by $\mathcal{Q}_{d,p,\beta}$ the set of quadratic forms $[a, b, c] \in \mathcal{Q}_{d,p}$ such that $b \equiv \beta \pmod{2p}$. For every positive integer m , let $\varphi_m(j_p^*)$ be a unique polynomial of j_p^* satisfying $\varphi_m(j_p^*(\tau)) = q^{-m} + O(q)$. We define two modular trace functions:

$$\begin{aligned} \mathbf{t}_m^{(p)}(d) &:= \sum_{Q \in \mathcal{Q}_{d,p,\beta}/\Gamma_0(p)} \frac{1}{|\Gamma_0(p)_Q|} \varphi_m(j_p^*(\alpha_Q)), \\ \mathbf{t}_m^{(p^*)}(d) &:= \sum_{Q \in \mathcal{Q}_{d,p}/\Gamma_0^*(p)} \frac{1}{|\Gamma_0^*(p)_Q|} \varphi_m(j_p^*(\alpha_Q)), \end{aligned}$$

where α_Q is the root of $Q(X, 1) = 0$ in \mathfrak{H} . The definition of $\mathbf{t}_m^{(p)}(d)$ is independent of β . In addition, we set $\mathbf{t}_2^{(2^*)}(0) := 5$, $\mathbf{t}_2^{(3^*)}(0) = \mathbf{t}_2^{(5^*)}(0) := 3$, $\mathbf{t}_2^{(p^*)}(-1) := -1$, $\mathbf{t}_2^{(p^*)}(-4) := -2$, $\mathbf{t}_2^{(p^*)}(d) := 0$ for $d < -4$ or $-d \not\equiv \text{square} \pmod{4p}$ ($p = 2, 3, 5$). For the relation between two modular trace functions, see [6].

Remark 2.2. For $p = 1$, we put $j_1(\tau) := j(\tau) - 744 = \{(\eta(\tau)/\eta(2\tau))^8 + 2^8(\eta(2\tau)/\eta(\tau))^{16}\}^3 - 744$ and $\mathbf{t}_m(d) := \mathbf{t}_m^{(1)}(d)$.

3 Proof of Theorem 1.1

We give the proof only for the case $p = 3$; the other cases are proved in the same way.

Definition 3.1. For every positive integer t , we define the operator U_t by

$$\left(\sum a_n q^n \right) \Big| U_t := \sum a_{tn} q^n.$$

Then U_t sends a modular form to a modular form of the same weight. To prove Theorem 1.1, we need the following theorem, which is a special case $f = \varphi_m(j_p^*(\tau))$ of Theorem 1.1 in [7].

Theorem 3.2. The function

$$g_m^{(p^*)}(\tau) := \sum_{d>0} \mathbf{t}_m^{(p^*)}(d) q^d + (\sigma_1(m) + p\sigma_1(m/p)) - \sum_{k|m} k q^{-k^2}$$

(where $\sigma_1(x) = 0$ if $x \notin \mathbb{Z}$) is a meromorphic modular form of weight $3/2$, holomorphic outside the cusps, with respect to $\Gamma_0(4p)$, that is,

$$g_m^{(p^*)}(\tau) \in M_{3/2}^{mer}(\Gamma_0(4p)).$$

Here $M_k^{mer}(\Gamma)$ denotes the space of meromorphic modular forms of weight k with respect to Γ .

We prove Theorem 1.1. For the modular form $f(\tau) = \sum a_n q^n$, we define the functions \tilde{f}_0 , \tilde{f}_1 and \tilde{f}_2 by

$$\begin{aligned} \tilde{f}_0(\tau) &:= \frac{1}{3} \left\{ f(\tau) + f\left(\tau + \frac{1}{3}\right) + f\left(\tau + \frac{2}{3}\right) \right\}, \\ \tilde{f}_1(\tau) &:= \frac{1}{3} \left\{ f(\tau) + \zeta^{-1} f\left(\tau + \frac{1}{3}\right) + \zeta f\left(\tau + \frac{2}{3}\right) \right\}, \\ \tilde{f}_2(\tau) &:= \frac{1}{3} \left\{ f(\tau) + \zeta f\left(\tau + \frac{1}{3}\right) + \zeta^{-1} f\left(\tau + \frac{2}{3}\right) \right\} \end{aligned}$$

where $\zeta = e^{2\pi i/3}$. For each $k \pmod{3}$, then \tilde{f}_k has a Fourier expansion of the form $\tilde{f}_k(\tau) = \sum_{n \equiv k(3)} a_n q^n$, and it is also a modular form of the same weight. By Theorem 3.2, we have

$$g_2^{(3^*)}(\tau) = \sum_{d=-4}^{\infty} \mathbf{t}_2^{(3^*)}(d) q^d \in M_{3/2}^{mer}(\Gamma_0(12)).$$

Now consider the modular form $g_2^{(3*)}(\tau) \cdot \theta_0(\tau)$ where $\theta_0(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2} \in M_{1/2}(\Gamma_0(4))$. This form is of weight 2 and we have

$$g_2^{(3*)}(\tau) \cdot \theta_0(\tau) = \left(\sum_{d=-4}^{\infty} \mathbf{t}_2^{(3*)}(d) q^d \right) \cdot \left(\sum_{r \in \mathbb{Z}} q^{r^2} \right) = \sum_{n=-4}^{\infty} \left(\sum_{r \in \mathbb{Z}} \mathbf{t}_2^{(3*)}(n - r^2) \right) q^n \in M_2^{mer}(\Gamma_0(12)).$$

Similarly, the product $g_2^{(3*)}(\tau) \cdot \theta_0(9\tau)$ is also a modular form of weight 2 and its Fourier expansion is

$$g_2^{(3*)}(\tau) \cdot \theta_0(9\tau) = \sum_{n=-4}^{\infty} \left(\sum_{r \in \mathbb{Z}} \mathbf{t}_2^{(3*)}(n - (3r)^2) \right) q^n = \sum_{n=-4}^{\infty} \left(\sum_{r \equiv 0(3)} \mathbf{t}_2^{(3*)}(n - r^2) \right) q^n \in M_2^{mer}(\Gamma_0(36)).$$

We put

$$\begin{aligned} F(\tau) &:= \left(g_2^{(3*)}(\tau) \cdot \theta_0(\tau) \right) \Big|_{U_4} = \sum_{n=-1}^{\infty} \left(\sum_{r \in \mathbb{Z}} \mathbf{t}_2^{(3*)}(4n - r^2) \right) q^n \in M_2^{mer}(\Gamma(12)), \\ G(\tau) &:= \left(g_2^{(3*)}(\tau) \cdot \theta_0(9\tau) \right) \Big|_{U_4} = \sum_{n=-1}^{\infty} \left(\sum_{r \equiv 0(3)} \mathbf{t}_2^{(3*)}(4n - r^2) \right) q^n \in M_2^{mer}(\Gamma(36)). \end{aligned}$$

Then $F(\tau)$ and $G(\tau)$ are meromorphic modular forms of weight 2. Moreover, for

$$\begin{aligned} j'_3(\tau) &= \sum_{n=-1}^{\infty} n c_n^{(3)} q^n \in M_2^{mer}(\Gamma_0(3)), \\ E_2^{(3)}(\tau) &:= \frac{1}{2} (3E_2(3\tau) - E_2(\tau)) = 1 + 12 \sum_{n=1}^{\infty} \sigma_1^{(3)}(n) q^n \in M_2(\Gamma_0(3)), \end{aligned}$$

(where the prime denotes $(2\pi i)^{-1} d/d\tau$ and $E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$ is the Eisenstein series of weight 2), we put

$$H(\tau) := j'_3(\tau) - \frac{3}{2} E_2^{(3)}(\tau) = -\frac{1}{q} - \frac{3}{2} + \sum_{n=1}^{\infty} (n c_n^{(3)} - 18 \sigma_1^{(3)}(n)) q^n \in M_2^{mer}(\Gamma_0(3)).$$

Then, the theorem in the case of $p = 3$ is equivalent to the following identities of modular forms:

$$2\tilde{H}_0(\tau) = -\tilde{G}_0(\tau), \quad 2\tilde{H}_1(\tau) = \tilde{F}_1(\tau), \quad 2\tilde{H}_2(\tau) = \tilde{F}_2(\tau).$$

Since these modular forms are of weight 2 on $\Gamma(36)$, we see that, by the Riemann-Roch theorem, it is enough to check the coincidence of Fourier coefficients on both sides of the equalities up to q^{3960} . We checked this by using Mathematica and Pari-GP.

Similarly, we can show the equation $j_3^*(\tau) = j_3(\tau) - 3(j_3|U_3)(\tau)$, and we obtain $c_n^{(3*)} = c_n^{(3)} - 3c_{3n}^{(3)}$.

4 Proof of Theorem 1.3

In this section, we give an overview of the proof. Since we can prove any case in the same way as [4], we give the proof only for the case $p = 3$. First, we prepare for the proof.

Definition 4.1. *The binary quadratic forms*

$$\left\{ \begin{array}{ll} [3, 0, d/12] & (-d \equiv 0 \pmod{12}), \\ [3, 1, (d+1)/12], [3, -1, (d+1)/12] & (-d \equiv 1 \pmod{12}), \\ [3, 2, (d+4)/12], [3, -2, (d+4)/12] & (-d \equiv 4 \pmod{12}), \\ [3, 3, (d+9)/12] & (-d \equiv 9 \pmod{12}) \end{array} \right.$$

are forms with discriminant $-d$ and are called the principal form of discriminant $-d$.

Lemma 4.2. *The following conditions are equivalent for a form $Q \in \mathcal{Q}_{d,3}$:*

- (1) *There are $x, y \in \mathbb{Z}$ such that $Q(x, y) = 3$.*
- (2) *Q is $\Gamma_0^*(3)$ -equivalent to $[3, B, C]$ for some $B, C \in \mathbb{Z}$.*
- (3) *Q is $\Gamma_0^*(3)$ -equivalent to a principal form of discriminant $-d$.*

This lemma can be proved in the same way as Lemma 2.2 in [4]. The key theorem for the proof of Theorem 1.3 is the following.

Theorem 4.3. *(Laplace's method). Suppose that $h(t)$ is a real-valued C^2 -function defined on the interval (a, b) (with $a, b \in \mathbb{R}$). If we further suppose that h has a unique maximum at $t = c$ with $a < c < b$ so that $h'(c) = 0$ and $h''(c) < 0$, then, we have*

$$\int_a^b e^{\lambda h(t)} dt \sim e^{\lambda h(c)} \left(\frac{-2\pi}{\lambda h''(c)} \right)^{1/2}$$

as $\lambda \rightarrow \infty$.

We prove Theorem 1.3. By definition,

$$\mathbf{t}_2^{(3*)}(d) := \sum_{Q \in \mathcal{Q}_{d,3}/\Gamma_0^*(3)} \frac{1}{|\Gamma_0^*(3)_Q|} \varphi_2(j_3^*(\alpha_Q)).$$

If $Q = [a, b, c]$ is the element of $\mathcal{Q}_{d,3}$, we have

$$e^{2\pi i \alpha_Q} = \exp\left(2\pi i \left(\frac{-b + i\sqrt{d}}{2a}\right)\right) = \exp\left(-\frac{\pi i b}{a}\right) \exp\left(-\frac{\pi \sqrt{d}}{a}\right)$$

and consequently;

$$\begin{aligned} \varphi_2(j_3^*(\alpha_Q)) &= q^{-2} + O(q) \\ &= \exp\left(\frac{2\pi i b}{a}\right) \exp\left(\frac{2\pi \sqrt{d}}{a}\right) + O\left(\exp\left(-\frac{\pi \sqrt{d}}{a}\right)\right). \end{aligned}$$

By this calculation, the contribution to $\mathbf{t}_2^{(3*)}(d)$ comes only from classes of forms with $a = 3$. By Lemma 4.2, any such form is equivalent to a principal form, so that we have

$$\mathbf{t}_2^{(3*)}(d) = O\left(\exp\left(-\frac{\pi \sqrt{d}}{3}\right)\right) + \exp\left(\frac{2\pi \sqrt{d}}{3}\right) \times \begin{cases} 1 & (d \equiv 0, 3 \pmod{12}), \\ -1 & (d \equiv 8, 11 \pmod{12}). \end{cases}$$

Combining this formula with Theorem 1.1, we obtain

$$c_n^{(3)} \sim \frac{1}{2n} \times \begin{cases} -\sum_{\substack{r \equiv 0(3) \\ 4n \geq r^2}} \exp(2\pi \sqrt{4n - r^2}/3) & (n \equiv 0 \pmod{3}), \\ \sum_{\substack{r \equiv 1, 2(3) \\ 4n \geq r^2}} \exp(2\pi \sqrt{4n - r^2}/3) & (n \equiv 1 \pmod{3}), \\ -\sum_{\substack{r \equiv 0(3) \\ 4n \geq r^2}} \exp(2\pi \sqrt{4n - r^2}/3) & (n \equiv 2 \pmod{3}). \end{cases}$$

For each $k = 0, 1, 2$, we consider the sum

$$S_n^{(k)} := \frac{3}{2\sqrt{n}} \sum_{\substack{r \equiv k(3) \\ 4n \geq r^2}} e^{\frac{4}{3}\pi \sqrt{n} \sqrt{1 - \frac{r^2}{4n}}} = \frac{3}{2\sqrt{n}} \sum_{\substack{l \in \mathbb{Z} \\ 4n \geq (3l+k)^2}} e^{\frac{4}{3}\pi \sqrt{n} \sqrt{1 - \frac{(3l+k)^2}{4n}}},$$

and view this sum as a Riemann sum for the function $t \mapsto e^{4\pi \sqrt{n} \sqrt{1-t^2}/3} : (-1, 1) \rightarrow \mathbb{R}$. We can show that $S_n^{(k)}$ is asymptotic to the corresponding Riemann integral J_n where

$$J_n := \int_{-1}^1 e^{4\pi \sqrt{n} \sqrt{1-t^2}/3} dt.$$

(For further detail, see [4]). Moreover, applying Laplace's method to the case $\lambda = \sqrt{n}$ and $h(t) = 4\pi\sqrt{1-t^2}/3$ on $(-1, 1)$, we have

$$J_n \sim e^{\sqrt{n} \cdot 4\pi/3} \cdot \left(\frac{-2\pi}{-4\pi\sqrt{n}/3} \right)^{1/2} = \frac{\sqrt{3}}{\sqrt{2}n^{1/4}} e^{4\pi\sqrt{n}/3}.$$

Putting these asymptotic formulas together, we obtain

$$\begin{aligned} c_n^{(3)} &\sim \frac{1}{3\sqrt{n}} \times \begin{cases} -S_n^{(0)} & (n \equiv 0 \pmod{3}), \\ S_n^{(1)} + S_n^{(2)} & (n \equiv 1 \pmod{3}), \\ -S_n^{(0)} & (n \equiv 2 \pmod{3}), \end{cases} \\ &\sim \frac{e^{4\pi\sqrt{n}/3}}{\sqrt{6}n^{3/4}} \times \begin{cases} -1 & (n \equiv 0 \pmod{3}), \\ 2 & (n \equiv 1 \pmod{3}), \\ -1 & (n \equiv 2 \pmod{3}) \end{cases} \end{aligned}$$

as $n \rightarrow \infty$.

5 Tables of $\mathbf{t}_m^{(p^*)}(d)$ and $\mathbf{t}_m^{(p)}(d)$ ($-4 \leq d \leq 50$)

d	$\mathbf{t}_1^{(2^*)}(d)$	$\mathbf{t}_2^{(2^*)}(d)$	$\mathbf{t}_1^{(2)}(d)$	$\mathbf{t}_2^{(2)}(d)$
-4	0	-2	0	-4
-1	-1	-1	-1	-1
0	1	5	2	10
4	-26	518	-52	1036
7	-23	-8215	-23	-8215
8	76	7180	152	14360
12	-248	52760	-496	105520
15	-1	-385025	-1	-385025
16	518	287710	1036	575420
20	-1128	1263640	-2256	2527280
23	-94	-6987870	-94	-6987870
24	2200	4831256	4400	9662512
28	-4096	16572370	-8192	33144740
31	93	-78987171	93	-78987171
32	7180	52263100	14360	104526200
36	-12418	153553438	-24836	307106876
39	-236	-663068908	-236	-663068908
40	20632	425670680	41264	851341360
44	-33512	1122593352	-67024	2245186704
47	235	-4515675925	235	-4515675925
48	53256	2835914280	106512	5671828560

d	$\mathbf{t}_1^{(3^*)}(d)$	$\mathbf{t}_2^{(3^*)}(d)$	$\mathbf{t}_1^{(3)}(d)$	$\mathbf{t}_2^{(3)}(d)$
-4	0	-2	0	-2
-1	-1	-1	-1	-1
0	1	3	2	6
3	-7	33	-14	66
8	-34	-410	-34	-410
11	22	-1082	22	-1082
12	26	1428	52	2856
15	-69	3195	-138	6390
20	-116	-11892	-116	-11892
23	115	-22797	115	-22797
24	174	28710	348	57420
27	-241	53223	-482	106446
32	-410	-140222	-410	-140222
35	492	-240500	492	-240500
36	492	287244	984	574488
39	-705	477567	-1410	955134
44	-1060	-1081096	-1060	-1081096
47	1272	-1718792	1272	-1718792
48	1442	2004918	2884	4009836

d	$\mathbf{t}_1^{(5*)}(d)$	$\mathbf{t}_2^{(5*)}(d)$	$\mathbf{t}_1^{(5)}(d)$	$\mathbf{t}_2^{(5)}(d)$
-4	0	-2	0	-2
-1	-1	-1	-1	-1
0	1	3	2	6
4	-8	-6	-8	-6
11	-12	-124	-12	-124
15	-19	93	-38	186
16	-6	-270	-6	-270
19	20	132	20	132
20	6	268	12	536
24	-44	216	-44	216
31	-39	-1863	-39	-1863
35	-44	1668	-88	3336
36	20	-3054	20	-3054
39	53	1653	53	1653
40	56	2868	112	5736
44	-136	2416	-136	2416

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